Critical exponents for a three-dimensional O(n)-symmetric model with $n > 3$

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Critical exponents for the three-dimensional O(n)-symmetric model with $n > 3$ are estimated on the basis of six-loop renormalization-group (RG) expansions. A simple Padé-Borel technique is used for the resummation of the RG series and the Padé approximants $[L/1]$ are shown to give rather good numerical results for all calculated quantities. For large n, the fixed point location g_c and the critical exponents are also determined directly from six-loop expansions without addressing the resummation procedure. An analysis of the numbers obtained shows that resummation becomes unnecessary when n exceeds 28 provided an accuracy of about 0.01 is adopted as satisfactory for g_c and the critical exponents. Further, results of the calculations performed are used to estimate the numerical accuracy of the $\frac{1}{n}$ expansion. The same value $n = 28$ is shown to play the role of the lower boundary of the domain where this approximation provides high-precision estimates for the critical exponents.

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I. INTRODUCTION

The field-theoretical three-dimensional O(n)-symmetric model with self-interaction of $\lambda\varphi^4$ type is known to describe the critical behavior of many basic physical systems such as Ising ($n = 1$) and Heisenberg ($n = 3$) ferromagnets, superfluid Bose liquids ($n = 2$), polymers ($n = 0$), etc. Nickel, Meiron, and Baker, Jr. calculated all two-point and four-point Feynman graphs for this model up to a six-loop order [1], paving the way for obtaining perturbative expansions of unprecedented length for β function and critical exponents. These expansions were then explicitly found and used, being resummed in various manners, to estimate the stable fixed point coordinate and numerical values of critical exponents [2–4]. The values obtained are referred to today as the most accurate (canonical) numbers [5].

Explicit expressions for the renormalization-group (RG) functions and numerical estimates were presented in Refs. [2–4] only for $n = 0, 1, 2, 3$. At the same time, it is desirable to have such results for $n > 3$. They are interesting from, at least, three points of view. First, there are numerous physical systems with many-component order parameters and these results may be relevant to their critical or effective critical behavior (see, e.g., Refs. [6–8]). Second, such calculations would enable one to clear up where resummation procedures applied to the RG series become unnecessary, i.e., how large are the values of n for which the theory may be thought of as possessing a small parameter. And third, high-precision numerical estimates of critical exponents for $n \gg 1$, when compared with their counterparts given by $\frac{1}{n}$ expansion, would provide information about the numerical accuracy of this familiar approximation scheme.

Below, the six-loop perturbative expansions for β function and critical exponents η and γ (γ^{-1}) are calculated for arbitrary n. The fixed point coordinate g_c and critical exponents are estimated on the base of Padé-Borel resummation procedure and a comparison of these numbers with those given by the unresummed RG series and a $\frac{1}{n}$ expansion is made. The outline of the paper is as follows. In Sec. II the renormalization scheme is formulated, RG expansions are written down, the resummation technique is described, and numerical results obtained are collected. In Sec. III, they are discussed along with their analogs resulting from unresummed six-loop series and a $\frac{1}{n}$ expansion, and corresponding inferences are presented. Section IV contains conclusions.

II. RENORMALIZATION-GROUP SERIES AND NUMERICAL RESULTS

The Hamiltonian of the model to be studied reads

$$H = \frac{1}{2}\int d^3x\Big[(\nabla\varphi_\alpha)^2 + m_0^2\varphi_\alpha^2 + \frac{2}{4!}\lambda(\varphi_\alpha^2)^2\Big] \quad , \tag{2.1}$$

where φ_α is a vector order parameter field, $\alpha = 1, \ldots, n$, a bare mass squared, m_0^2 is proportional to the deviation from the mean-field transition point.

We calculate the β function and critical exponents within a massive theory. The renormalized Green function $G_R(p, m, g)$ and a four-point vertex function $\Gamma_R(p, m, g)$ are normalized at zero momenta in a conventional way:

$$\begin{aligned} &G_R^{-1}(0, m, g) = m^2 \quad , \\ &\frac{\partial G_R^{-1}(p, m, g)}{\partial p^2}\bigg|_{p^2=0} = 1 \quad , \\ &\Gamma_R(0, m, g) = mg \quad , \end{aligned} \tag{2.2}$$

with one extra condition imposed on the φ^2 insertion:

$$\Gamma_R^{1,2}(p, q, m, g)\Big|_{p=q=0} = 1 \quad . \tag{2.3}$$

Since combinatorial factors and momentum integrals for two-point and four-point Feynman graphs are known [1], the calculation of the β function and critical exponents (anomalous dimensions) within a six-loop approximation is straightforward (see, e.g., [9]). The results are as follows:

$$\beta(g) = g - g^2 + \frac{1}{(n+8)^2}\Big(6.074\,074\,08n + 28.148\,148\,15\Big)g^3 - \frac{1}{(n+8)^3}\Big(1.348\,942\,76n^2 + 54.940\,376\,98n + 199.640\,417\,0\Big)g^4 + \frac{1}{(n+8)^4}\Big(-0.155\,645\,89n^3 + 35.820\,203\,78n^2 + 602.521\,230\,5n + 1832.206\,732\Big)g^5 - \frac{1}{(n+8)^5}\Big(0.051\,236\,18n^4 + 3.237\,876\,20n^3 + 668.554\,336\,8n^2 + 7819.564\,764n + 20\,770.176\,97\Big)g^6 + \frac{1}{(n+8)^6}\Big(-0.023\,424\,17n^5 - 1.071\,798\,39n^4 + 265.835\,703\,2n^3 + 12\,669.221\,19n^2 + 114\,181.4357n + 271\,300.0372\Big)g^7, \tag{2.4}$$

$$\eta(g) = \frac{1}{(n+8)^2}\Big(0.296\,296\,296\,3n + 0.592\,592\,592\,6\Big)g^2 + \frac{1}{(n+8)^3}\Big(0.024\,684\,001\,4n^2 + 0.246\,840\,014n + 0.394\,944\,022\,4\Big)g^3 + \frac{1}{(n+8)^4}\Big(-0.004\,298\,562\,6n^3 + 0.667\,985\,920\,2n^2 + 4.609\,221\,057n + 6.512\,109\,933\Big)g^4 - \frac{1}{(n+8)^5}\Big(0.006\,550\,922\,2n^4 - 0.132\,451\,061\,4n^3 + 1.891\,139\,282n^2 + 15.188\,093\,40n + 21.647\,206\,43\Big)g^5 + \frac{1}{(n+8)^6}\Big(-0.005\,548\,920\,2n^5 - 0.020\,399\,448\,5n^4 + 3.054\,030\,987n^3 + 64.077\,446\,56n^2 + 300.720\,893\,3n + 369.713\,073\,9\Big)g^6 \;, \tag{2.5}$$

$$\gamma^{-1}(g) = 1 - \frac{n+2}{2(n+8)}g + \frac{n+2}{(n+8)^2}g^2 - \frac{1}{(n+8)^3}\Big(0.879\,558\,892\,6n^2 + 6.485\,476\,868n + 9.452\,718\,166\Big)g^3 + \frac{1}{(n+8)^4}\Big(-0.128\,332\,104\,3n^3 + 7.966\,740\,703n^2 + 51.844\,212\,98n + 70.794\,806\,31\Big)g^4 - \frac{1}{(n+8)^5}\Big(0.049\,096\,605\,8n^4 + 4.288\,152\,493n^3 + 108.361\,821\,9n^2 + 537.813\,610\,5n + 675.699\,607\,7\Big)g^5 + \frac{1}{(n+8)^6}\Big(-0.025\,926\,794\,5n^5 - 1.618\,627\,843n^4 + 85.545\,697\,46n^3 + 1538.818\,235n^2 + 6653.956\,526n + 7862.074\,086\Big)g^6 \;. \tag{2.6}$$

These series are known to be divergent (asymptotic). To extract the physical information which they contain, some resummation procedure should be employed. We use the Padé-Borel method, i.e., construct Padé approximants $[L/M]$ for Borel transforms which are related to functions to be found ("sum of series") by the formula

$$f(x) = \sum_{k=0}^{\infty} c_k x^k = \int_0^{\infty} e^{-t}F(xt)dt \;, \tag{2.7}$$

$$F(y) = \sum_{k=0}^{\infty} \frac{c_k}{k!}y^k \;, \tag{2.8}$$

and then evaluate the integral (2.7), where series (2.8) possessing nonzero radii of convergence are replaced by corresponding Padé approximants.

Starting from the six-loop expansions available, it is possible to construct different sets of Padé approximants: $[L/1]$, $[L-1/2]$, etc., where $L = 6$ for β function and $L = 5$ for critical exponents. As we have found, approximants

$$[L/1] = (1 + b_1 y)^{-1}\sum_{i=0}^{L} a_i y^i, \tag{2.9}$$

which generate the following expressions for sums of the series:

$$f(x) = ze^{-z}\mathrm{Ei}(z)\sum_{i=0}^{L} a_i(-b_1)^{-i} - \sum_{i=1}^{L} a_i(-b_1)^{-i}\sum_{k=0}^{i-1} k!z^{-k},$$
$$z = -\frac{1}{b_1 x} \;, \qquad \mathrm{Ei}(z) = \int_{-\infty}^{z} e^t t^{-1}dt, \tag{2.10}$$

TABLE I. The stable fixed point location and the critical exponents obtained within the six-loop approximation using the Padé-Borel resummation technique.

n	g_c	γ	η	ν	α	β
0	1.402	1.160	0.034	0.589	0.231	0.305
	1.421[a]	1.161[a]	0.026[a]	0.588	0.236	0.302
	1.417[b]	1.162[b]	0.026[b]	0.588[b]		0.302[b]
1	1.401	1.239	0.038	0.631	0.107	0.327
	1.416[a]	1.241[a]	0.031[a]	0.630[a]	0.110[a]	0.324[a]
	1.414[b]	1.240[b]	0.032[b]	0.630[b]		0.325[b]
			0.035[c]	0.628[c]		
2	1.394	1.315	0.039	0.670	- 0.010	0.348
	1.406[a]	1.316[a]	0.032[a]	0.669[a]	-0.007[a]	0.346[a]
	1.405[b]	1.316[b]	0.034[b]	0.669[b]		0.346[b]
			0.0374[c]	0.665[c]		
3	1.383	1.386	0.038	0.706	- 0.117	0.366
	1.392[a]	1.390[a]	0.031[a]	0.705[a]	- 0.115[a]	0.362[a]
	1.391[b]	1.387[b]	0.034[b]	0.705[b]		0.365[b]
			0.037[c]	0.698[c]		
4	1.369	1.449	0.036	0.738	- 0.213	0.382
5	1.353	1.506	0.034	0.766	- 0.297	0.396
6	1.336	1.556	0.031	0.790	- 0.370	0.407
7	1.319	1.599	0.029	0.811	- 0.434	0.417
8	1.303	1.637	0.027	0.830	- 0.489	0.426
9	1.288	1.669	0.025	0.845	- 0.536	0.433
10	1.274	1.697	0.024	0.859	- 0.576	0.440
12	1.248	1.743	0.021	0.881	- 0.643	0.450
14	1.226	1.779	0.019	0.898	- 0.693	0.457
16	1.207	1.807	0.017	0.911	- 0.732	0.463
18	1.191	1.829	0.015	0.921	- 0.764	0.468
20	1.177	1.847	0.014	0.930	- 0.789	0.471
24	1.154	1.874	0.012	0.942	- 0.827	0.477
28	1.136	1.893	0.010	0.951	- 0.854	0.481
32	1.122	1.908	0.009	0.958	- 0.875	0.483

[a]Quoted from Ref. [3]

[b]Quoted from Ref. [2].

[c]Quoted from Ref. [10].

give the best results. They are presented in Table I. The estimates for γ and η originate from series (2.5) and (2.6), while numerical values of critical exponents ν, α, and β were determined by means of well-known scaling relations. The exponent γ was calculated also via resummed RG expansion for the exponent $\eta_2 = (1-\gamma)(2-\eta)/\gamma$ and numbers were obtained which differ from those resulting from (2.6) by no more than 0.003; corresponding averages stand in Table I. This table contains as well, for comparison, numerical results found earlier for $n = 0, 1, 2, 3$ on the base of higher-order RG expansions in 3 and $4-\epsilon$ dimensions using alternative resummation techniques [2, 3, 10]. It is worth discussing these results along with ours in more detail.

As we can see, there are small differences between our estimates and their counterparts obtained in Refs. [2, 3] from the three-dimensional RG expansions of the same length. They are caused by use of different resummation procedures. Indeed, the authors of Refs. [2, 3] employed the Borel-Leroy transformation,

$$f(x) = \int_0^\infty t^B e^{-t} \mathcal{F}(xt) dt, \tag{2.11}$$

instead of Eq. (2.7) in their calculations. The parameter B was chosen to meet the known large-order behavior of coefficients c_k in perturbative expansions [11, 12]:

$$c_k \sim k!(-a)^k k^b \quad , \qquad k \to \infty \quad , \tag{2.12}$$

where $a = 0.147\ 774$ for the model (2.1) and b is equal to $2 + \frac{n}{2}$, or $3 + \frac{n}{2}$, or $5 + \frac{n}{2}$ depending on the RG function expanded. We use a much simpler method which ignores some part of information (2.12) but leads, nevertheless, to numerical results rather close to those given by more sophisticated techniques. It is not surprising since the main property of c_k — their factorial growth, is taken into account in our analysis, while the rest of information about c_k being incorporated enables one to reduce the apparent errors of estimation keeping the location of the fixed point and critical exponents practically unchanged (see, e.g., Ref. [3] for detail). Dealing with simple Padé approximants $[L/1]$, we avoid also, to a certain extent, the problem of poles. The point is that these approximants turn out to have no real and positive poles for $n < 38$ in the case of critical exponents and up to $n = 80$ for the β function. That is why Table I ends at $n = 32$. Since for $n = 0, 1, 2, 3$ our procedure gives critical exponents values which are almost identical to the known high-precision estimates [2, 3, 10], we believe that

TABLE II. Coordinates of the fixed point obtained from Eq. (2.4) with use of the Padé-Borel resummation procedure (PB) and by direct summation (DS).

n		20	24	28	32	36	40	50	60
g_c	(DS)	1.2184	1.1725	1.1458	1.1273	1.1134	1.1025	1.0830	1.0699
	(PB)	1.1768	1.1538	1.1359	1.1216	1.1099	1.1003	1.0822	1.0696

TABLE III. Values of the critical exponent γ obtained by the direct summation of the RG expansion (DS), by means of Padé-Borel-resummation technique (PB) and from the $\frac{1}{n}$ expansion [Eq. (3.1)].

n	20	24	28	32	36	40	50	70	100	500
(DS)	1.8990	1.8991	1.9075	1.9165	1.9245	1.9314	1.9447	1.9606	1.9725	1.9946
(PB)	1.8466	1.8737	1.8932	1.9078	1.9222					
$\frac{1}{n}$	1.8702	1.8930	1.9090	1.9208	1.9299	1.9372	1.9501	1.9646	1.9754	1.9951

the rest of the results listed in this table are also very close to exact numbers.

III. LARGE n AND $\frac{1}{n}$ EXPANSION

How can we estimate g_c and critical exponents for $n \gtrsim 30$? It is well known [and clearly seen from Eqs. (2.4)–(2.6)] that coefficients of RG expansions are decreasing when n grows up. Hence, for large enough n the theory should possess a true small parameter as, say, the quantum electrodynamics does. In such a case, all quantities of interest can be obtained directly from corresponding perturbative expansions, without addressing the resummation technique. To find the minimal value of n which may be referred to as "large enough" we have calculated g_c for $20 \le n \le 60$ using original and Padé-Borel-resummed series (2.4). (It should be remembered that the approximant [6/1] for the Borel transform of β function has no dangerous poles within this segment.) The results are presented in Table II. Values of g_c given by these two approximations are seen to differ from each other by 0.9% for $n = 28$ and this difference diminishes rapidly with increasing n. So, if the accuracy of order of 1% for g_c was adopted as satisfactory, the resummation of the six-loop expansion for the β function becomes unnecessary when n exceeds 28. The same turns out to be truth for the critical exponent γ as is seen from Table III (the first and the second lines).

For large n, another approximate method may be used to calculate critical exponents. We mean the famous $\frac{1}{n}$ expansion. Within the second order in $\frac{1}{n}$ exponents γ and η are known to be [13]

$$\gamma = 2 - \frac{24}{\pi^2}\frac{1}{n} + \frac{64}{\pi^4}\left(\frac{44}{9} - \pi^2\right)\frac{1}{n^2} \quad , \tag{3.1}$$

$$\eta = \frac{8}{3\pi^2}\frac{1}{n} - \frac{512}{27\pi^4}\frac{1}{n^2} \quad . \tag{3.2}$$

The series for other critical exponents are easily obtained via scaling relations.

It is interesting to evaluate the accuracy of numerical results given by $\frac{1}{n}$ expansion. We can get such information comparing numbers resulting from Eqs. (3.1) and (3.2) for various n with their counterparts obtained on the base of the resummed ($n \le 32$) and the unresummed ($n > 32$) six-loop RG series. On the other hand, this comparison would help us to determine the accuracy of the employed approximation itself in the limit $n \to \infty$, where $\frac{1}{n}$-expansion's results are exact.

Corresponding estimates for exponent γ are listed in Table III. These numbers show that numerical accuracy of Eq. (3.1) becomes better than 1% when n exceeds 28. Values of η given by the six-loop RG series and Eq. (3.2) are very small and not presented here. They differ from each other by approximately 10% for $n > 28$. Moreover, this discrepancy persists up to largest values of n studied. It is not a surprise. The point is that, for extremely large n, only leading terms in n contribute to η in each order in g. Since $g_c = 1 + O(n^{-1})$, g_c should be put equal to unity within this limit. Hence, corresponding total contribution in the case of the six-loop RG series may be found by summing the coefficients of all leading terms in Eq. (2.5). Such a procedure gives $\eta = \frac{0.30458}{n}$, while the exact asymptotic expression resulting from Eq. (3.2) is $\eta = \frac{0.27019}{n}$. So, the approximate asymptotic estimate for η differs from the exact one by 13%. This difference, however, practically does not influence numerical values of other critical exponents calculable by scaling relations since for $n > 28$, the exponent $\eta < 0.01$.

We see that simple formulas (3.1) and (3.2) enable one to estimate all critical exponents for the model (2.1) with an accuracy of order of 0.01 provided $n \ge 28$. Moreover, for such n second-order terms in these formulas may be, in fact, neglected since their contributions are very small.

IV. CONCLUSION

Critical exponents of the three-dimensional O(n)-symmetric model have been estimated from the six-loop RG series for $n > 3$. Renormalization-group expansions have been resummed by means of a simple Padé-Borel technique and approximants [6/1] (β function) and [5/1] (critical exponents) have been shown to provide rather good numerical results for all calculated quantities. It has been found that for $n \ge 28$, the theory may be thought as possessing a small parameter, i.e., the fixed point coordinate and critical exponents may be determined with errors about 0.01 or less directly from the higher-order RG series, without use of a resummation procedure. Numerical accuracy of the $\frac{1}{n}$ expansion has been also estimated. The same value, $n = 28$, has been shown to play a role of a lower boundary of the region where this approximation provides high-precision results for critical exponents.

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